SWANSEA SUMMER SCHOOL in NONLINEAR PDEs.

COURSE ON

FULLY NONLINEAR DEGENERATE ELLIPTIC EQUATIONS: QUALITATIVE PROPERTIES of VISCOSITY SOLUTIONS.

PLAN:

L.1: Definitions of VISCOSITY SOLUTIONS: motivations & basic properties

L.2: Comparison principles; STRONG maximum principles. (SMP).

L.3: SMP: examples.
Liouville properties: some classical results.

L.4: Liouville properties for viscosity SUB-SOLUTIONS.

REFERENCES: for lectures 1-2:

[CIL] Crandall - Ishii - P.L. Lions: Users' guide 1992

[BCD] M.B. - I. Capuzzo-Dolcetta: book on Hamilton-Jacobi eqs. Birkhäuser 1997

[CC] Caffarelli - Cabré: book AMS 1995

[K] S. Koike: A beginner's guide to viscosity solutions M.S.J. Memoir 2004

[GT] Gilbarg - Trudinger. book Springer 1983

On SUB-ELLIPTIC equations

[BLU] Bonfiglioli, Lanconelli, Uguzzoni, book Springer 2007

[M] J. Manfredi: lecture notes on FULLY NONLINEAR subelliptic equations 2003

REFERENCES for lectures 3-4.:

M.B. - ANNALISA CESARONI, J.D.E. 2016

M.B. - ALESSANDRO GOFFI: Calc. Var. PDE 2019 (on SMP)

- Math. Ann. 2022
- Adv. Diff. Eqs. 2023

LECTURE 1, July 1, 2024

Fully nonlinear 2^{nd} order PDE:

$$(E) \qquad F(x, u, Du, D^2u) = 0 \qquad \text{in } \Omega \subseteq \mathbb{R}^n \text{ open}$$

$F : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S_n \to \mathbb{R}$ CONTINUOUS

F DEGENERATE ELLIPTIC if

$$F(x, r, p, X) \leq F(x, r, p, Y) \qquad \forall\, X - Y \geq 0$$

NONINCREASING $(-\Delta u = 0)$ $\quad Y \leq X$

(E) or F is PROPER if, in addition

$F(x, r, p, X) \leq F(x, s, p, X)$ if $r \leq s$

NONDECREASING

MAIN BASIC Properties of (E) PROPER is a form of MAXIMUM PRINCIPLE :

if $u \in C^2(\Omega)$ $\quad F(x, u, Du, D^2u) \leq 0$
$\varphi \in C^2(\Omega)$ $\quad F(x, \varphi, D\varphi, D^2\varphi) > 0$ $\Big\}$ in Ω

then $u - \varphi$ cannot have a NONNEGAT. LOC INT. MAX. $x_0 \in \Omega$

Pf If $(u-\varphi)(x_0) \geq 0$ is a loc max $\Rightarrow$

$u(x_0) \geq \varphi(x_0)$, $\quad D(u-\varphi)(x_0) = 0$, $\quad D^2(u-\varphi)(x_0) \leq 0$

$D^2u(x_0) \leq D^2\varphi(x_0) \Rightarrow$

$$0 < F(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq F(x_0, u(x_0), Du(x_0), D^2u(x_0)) \leq 0$$

$\otimes$

$\blacksquare$

Examples • 1st order eqs.

$H(x, u, Du) = 0 \qquad H \in C$

$r \mapsto H(x, r, p)$ nondecr. $\Rightarrow$ PROPER

• semilinear

$$-\operatorname{tr}(A(x)D^2u) + f(x,u,Du) = 0$$

proper if $A(x) \geq 0$, $r \mapsto f(x,r,p)$ nondec.

$\Rightarrow$ proper.

• H-J-BELLMAN eqs. α = parameter.

$$L^\alpha u = -\operatorname{tr}(A^\alpha(x)D^2u) + b^\alpha(x)\cdot Du + c^\alpha(x)u - \ell^\alpha(x)$$

$$F[u] = \sup_\alpha L^\alpha u = 0 \quad \text{or} \quad G[u] = \inf_\alpha L^\alpha u = 0$$

L^α proper $\forall \alpha$ $\Rightarrow$ F & G are proper

OPTIMAL STOCHASTIC CONTROL

(ex. : PUCCI max ops. , Monge-Ampère ...)

• MANY GEOMETRIC EQUATIONS

• SUBELLIPTIC EQS. $\Omega \subseteq \mathbb{R}^n$

GIVEN $m (\leq n)$ vector fields $(X_1, \dots, X_m) = \mathcal{X}$

"Horizontal" ∇ of u is

$$(X_1 u, \dots, X_m u) = D_{\mathcal{X}} u$$

Horizontal Hessian

$$D^2_{\mathcal{X}} u = (X_i X_j u)_{i,j=1,\dots,m}$$

Sub. ell.

$$G(x, u, D_{\mathcal{X}} u, (D^2_{\mathcal{X}} u)^*) = 0$$

exerc.

$$\leadsto F(x, u, Du, D^2 u)$$

G proper $\Rightarrow$ F proper

e.g. SUBLAPLACIANS $\quad - \mathrm{tr}\, D^2_{\mathcal{X}} u$

DEFINITIONS of VISCOSITY SOLUTIONS

Rmk. Max. Princ. implies a COMPARISON PRINC:

$u \in C^2(\Omega)$, $F[u] \leq 0$ in Ω $\Rightarrow$

$$\left.\begin{array}{l} \forall \varphi \in C^2(\Omega) \ \forall B \subseteq \Omega \quad F[\varphi] > 0 \text{ in } B \\ \qquad\qquad\qquad\qquad\qquad u \leq \varphi \text{ on } \partial B \end{array}\right] \ (P)$$

$\Rightarrow \quad u \leq \varphi$ in B.

Def (0) $u \in USC(\Omega)$ is a VISC. SUBSOL. of (E)

if $\forall \varphi \in C^2(\Omega)$ $\forall B \subseteq \Omega$ $\quad F[\varphi] > 0$ in B

$u \leq \varphi$ on ∂B

$\Rightarrow \quad u \leq \varphi$ in B. $\qquad$ [Caffarelli ...]

Def. 1 [CIL] [K] (i) $u \in USC(\Omega)$ is a V. SUBSOL. if

$\forall \varphi \in C^2(\Omega)$ $\forall x_0 \in \Omega$ loc. max pt. of $u - \varphi$

$$F(\cdot, u, D\varphi, D^2\varphi)\big|_{x_0} \leq 0$$

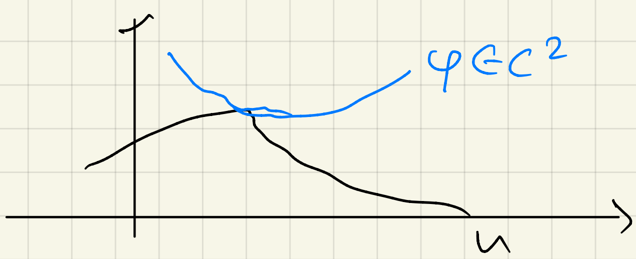
(ii) $u \in LSC(\Omega)$ is a V. SUPERSOL. if

$\forall \varphi \in C^2(\Omega)$ $\forall x_0 \in \Omega$ loc MIN. pt. of $u - \varphi$

$$F(\cdot, u, D\varphi, D^2\varphi)\big|_{x_0} \geq 0$$

(iii) $u \in C(\Omega)$ a VISC. SOL. is a SUB- & SUPERSOL.

RMK. it is equiv STRICT MAX & min of $u - \varphi$ & $u(x_0) = \varphi(x_0)$



RMKS • CONSISTENCY: u V. Sol. of E

u twice diff.le at x_0 $\Rightarrow$ $F(x_0, u(x_0), Du(x_0), D^2u(x_0)) = 0$

• MOTIVATIO OF NAME

(HJ) $\qquad u_t + H(x, D_x u) = 0$

$\varepsilon > 0$

$$(\varepsilon) \qquad u_t^\varepsilon + H(x, D_x u^\varepsilon) = \varepsilon \Delta_x u^\varepsilon$$

$\varepsilon_k \searrow 0 \quad u^{\varepsilon_k} \to u$ loc. unif

$\Rightarrow$ u is a viscosity sol. of (HJ)

Pf Exercice. Ex. STABILITY of VISCOSITY SOL.

CORNERSTONE RESULT. ! COMPARISON PRINCIPLES among SUB & SUPERSOLS.

For Dirichlet pbs.

(CP) $F[u] \le 0$, $F[v] \ge 0$ VISCO. SENSE

in Ω open $\quad u \le v$ on $\partial\Omega$ $\overset{?}{\Rightarrow}$ $u \le v$ in Ω.

N.B. (CP) $\Rightarrow$ UNIQUENESS of V. SOL. for

$$\begin{cases} F[u] = 0 & \text{in } \Omega \\ u = g \text{ given} & \text{on } \partial\Omega \end{cases}$$

Q: When is (CP) true? $\quad (F \equiv 0)$

A short history 1st order eqs.

- CRANDALL - EVANS - PL^2 1984

STRUCTURE $\qquad \delta u + H(x, Du) = 0 \qquad \delta > 0$

or $\qquad u_t + H(x, D_x u) = 0 \quad$ in $\underbrace{\mathbb{R}^n \times]0,T[}_{\Omega}$

(CP') $u \le v$ at $t=0 \Rightarrow u \le v$ in Ω.

if e.g.

$\omega(r)$ $\lim_{r \to 0^+} \omega(r) = 0$

(SH) $|H(x,p) - H(y,p)| \le \omega(|x-y|(1+|p|))$.

2nd order eqs. · R. Jensen 1988

$$\delta u + F(Du, D^2u) - f(x) = 0 \qquad \delta > 0$$

CIL : 89-90 $\delta u + F(x, Du, D^2u) = 0$

with "Lip type ass." in x ....

Ex: $F = -\mathrm{tr}(A(x) D^2u) + H(x, Du) = 0$

$\delta > 0$ $\delta u + F[\ ]$ sats. (CP) if

H sats (RH) & $A(x) = \sigma(x)\sigma^T(x)$

$\sigma \in$ Lip.

What about $\delta = 0$??

1 · R. Jensen (CP) is ok for $\delta = 0$ if F is UNIFORMLY ELLIPTIC : $\exists\ 0 < \lambda \le \Lambda$: $\forall\ N \ge 0$

$$\lambda \|N\| \le F(x, r, p, M-N) - F(x, r, p, M) \le \Lambda \|N\|$$

Ex. $F = -\mathrm{tr}(A(x) D^2u) + H(x, Du)$ (SL)

UN. ELL. $\Longleftrightarrow$ $\lambda \le \lambda_i(A(x)) \le \Lambda$ $\forall x$

eigenvalues

② U. ELL. for (SL) can be relaxed to
$A(x) \geq 0$ & $\exists j$! $a_{jj}(x) > 0$ $\forall x \in \bar{\Omega}$

e.g. SUBELL : $A(x) = \sigma(x)\sigma^T(x)$

$\sigma = (\sigma^1 \dots \sigma^m)$ (SP) ok. if $\exists j$
$\sigma^j_j(x) \neq 0$ $\forall x \in \Omega$.

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LECTURE 2

· STRONG MAX PRINCIPLE.

$\Omega \subseteq \mathbb{R}^n$ open CONNECTED

$Lu = -\operatorname{tr}(A(x)\, D^2 u) + b(x) \cdot Du + c(x) u$

Thm (Hopf 1927) $c \geq 0$, A unif. ell., A, b, c bdd. $u \in C^2(\Omega)$! $Lu \leq 0$.
If u has a non-loc. max at $x_0 \in \Omega \Rightarrow$
$u \equiv$ constant in Ω.

Rk Trivial $Lu < 0$ at x_0

$$\underbrace{-\operatorname{tr} A D^2 u}_{\geq 0} + \underset{0}{\underset{\shortparallel}{b \cdot Du}} + \underset{0}{\underset{\vee\!/}{cu}} \geq 0$$

$\boxtimes$

Consider $Lu = -\operatorname{tr}(A(x), D^2 u)$ $A(x) = \sigma(x)\sigma^T(x) \geq 0$

$\sigma = (\sigma^1, \dots, \sigma^m)$ $\sigma^j \in \mathbb{R}^n$ Lip.

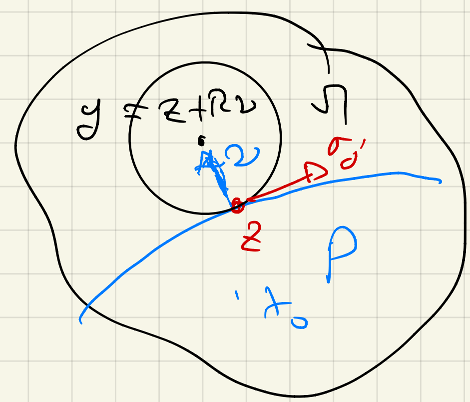
Let $u \in USC(\Omega)$ visc. subsol of $Lu \le 0$ in Ω.

Ass. $u(x_0) = M = \max_{\bar{\Omega}} u$

Propagation set $P = \{ x \in \Omega : u(x) = M \}$

If $P = \Omega$ SMP holds

Assume $\boxed{P \subsetneq \Omega}$



Def. ν = Bony ext. normal to P at $z \in \partial P$ if
$|\nu| = 1$

$$\exists R > 0 : \bar{B}(\underbrace{z + R\nu}_{y}, R) \cap \bar{P} = \{ z \}$$

LEMMA $(\sigma^j \cdot \nu)(z) = 0 \quad \forall j = 1, \dots, m$

All σ^j are tangential to P.

Rmk $\Rightarrow$ Hopf theorem if σ^j are a basis of $\mathbb{R}^n$

Pf of Lemma. By contradict. Ass $|\sigma^T \nu| > 0$

$$v(x) := e^{-\gamma R^2} - e^{-\gamma |x-y|^2} \qquad \gamma > 0$$

CLAIM : $Lv > 0$ in $B(z, r)$

$$Dv = 2\gamma e^{-\gamma |x-y|^2} (x - y)$$

$$D^2 v = 2\gamma e^{-\gamma |x-y|^2} \big(I - 2\gamma (\overset{-R\nu}{x-y}) \otimes (\overset{-R\nu}{x-y}) \big)$$

$$LV(z) = -\operatorname{tr}(\sigma\sigma^T \ \ ) = -2\gamma e^{-\gamma|z-y|^2}$$

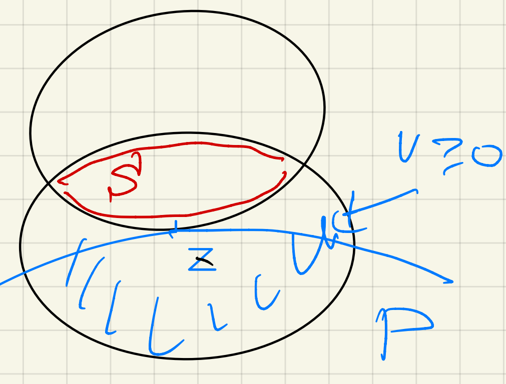
$$\big(A(z) - 2\gamma R^2 \underbrace{|\sigma^T \nu|^2}\big)$$

$$\operatorname{tr}(\sigma\sigma^T \nu \otimes \nu) \quad > 0$$

< 0 for γ large

$LV < 0$ in small ball $B(z,r)$

<u>Step 2</u> $\quad S = B(z,r) \cap B(y,R)$



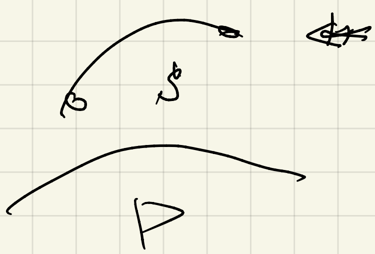
Claim $\quad u(x) - M \leq \varepsilon v(x)$ in S

sub sol. $\qquad$ super s

by Comparison Princ (CP)

because $\quad u(x) - M \leq 0 \qquad \partial\Omega$

$v \geq 0$



$u(x) - M < -\delta$

For ε small: $\quad u(x) - M \leq \varepsilon v(x)$

□ Claim.

<u>Step 3</u>

$u(x) - \varepsilon v(x) \leq M = u(z)$ in S

$=$ at z

$u - \varepsilon v$ has a max (in S) at z

$u - \varepsilon v$ has its max $B(z,r)$ at z

Def. VISCO SUBSOL: $\Longrightarrow$

$$L(\varepsilon V)(z) \leq 0 \qquad \otimes \qquad \text{Step } 1$$

$$\overset{\|}{\varepsilon L V} \qquad\qquad LV > 0$$

$\blacksquare$

Cor Hopf Thm. is true for $u \in USC$ visco subsol not necessarily C^2.

Tomorrow

Bdry MAX PRINCIPLE.